ON THE STABILITY OF SECONDARY TAYLOR FLOW BETWEEN ROTATING CYLINDERS WITH A WIDE GAP

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In a general formulation, the problem of the stability of small perturbations in a homogeneous viscous fluid between rotating cylinders with a wide gap is investigated numerically.

The problem of the stability limits of a laminar viscous flow between rotating cylinders has been investigated by many authors starting with Taylor. The calculations were carried out initially under the assumption that the cylinder radii are approximately identical, which in turn permitted approximation of the velocity distribution in the steady flow either by a constant or by a linear function. A survey of the methods used to solve the problem in the above-mentioned approximation is presented in [1], where the majority of solutions refer to the case of cylinder rotation in one direction. The initial Taylor method, consisting in the expansion of the solution in orthogonal functions and in obtaining the characteristic equation in the form of an infinite determinant is also elucidated in [1].

The ideas of the Taylor method were extended and developed in [2], in which the curvature of the unperturbed flow velocity profile was taken into account and numerical results were obtained for the ratio between the cylinder radii $R_1/R_2 = 1/2$ in a small range of variation of their angular velocities.

A thorough analysis of experimental investigations on this problem is presented in [3], in which graphs are given of the stability for experiments with both a small and a wide gap between the cylinders. In the latter case, a comparison with existing theoretical computations is presented [2]. It is noted that the experiments were conducted in a significantly broader range of variation of the cylinder angular velocities than the calculations, and that there is no satisfactory mathematical description of the complete viscous problem for the case of cylinders rotating oppositely.

It is hence expedient to use finite-difference methods to solve this problem. The papers [4, 5] can be noted in this area.

A neutral curve is computed in [4] probably for the outer cylinder at rest and the ratio $R_1/R_2 = 1/2$ (these data are not presented in the paper).

A great volume of calculational work to determine the stability curves for a broad range of variation in the ratio R_1/R_2 is presented in [5]. The problem of determining the eigenvalues with respect to the Reynolds criterion is solved in this paper by using the Runge–Kutta method on a finite-difference matrix equation approximating the initial system of six first order equations for the neutral perturbations with undetermined parameters for part of the boundary conditions.

We used the method of numerically determining several of the first eigenvectors and their corresponding eigenvalues, elucidated in [6], to solve this problem, where the eigenvalue problem is posed relative to the exponent λ in the exponential dependence of the solution on the time. The advantage of this method is that it permits obtaining a picture of the streamlines of the possible stable secondary flows directly, which are described by stream functions as components of the first eigenvector corresponding to the zero-th maximum eigenvalues of the problem.

1. Let us consider viscous fluid flow between cylinders of the radii R_1, R_2 rotating at the angular velocities Ω_1, Ω_2. The subscript 1 refers to the inner, and the subscript 2 to the outer cylinder. The

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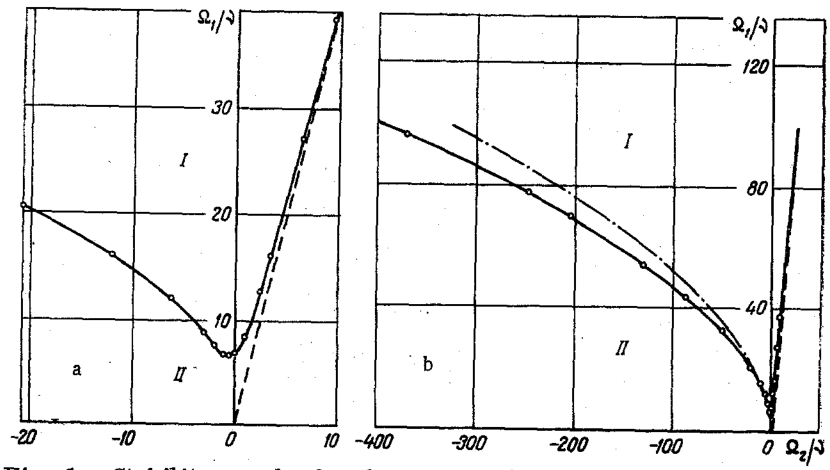

Fig. 1. Stability graphs for the ratio $R_2/R_1 = 2$ (a) of the cylinder radii: the dashed line corresponds to the Rayleigh stability criterion for the inviscid case $\Omega_1/\Omega_2 = R_2^2/R_1^2 = 4$, the dash-dot line corresponds to stability according to the experimental results of Donelly [3]; and the points correspond to the computation; I is instability and II is stability (b is the same but in a larger scale).

cylinders are not bounded along the z axis. The possible secondary Taylor flows are assumed periodic along the z axis with the wave parameter M and independent of the angle. Then the investigation of the stability of these flows in a linear approximation (see [1], Chapter 2) reduces to solving a system for the small perturbation amplitudes of the stream function $\psi(\tau, r)$ and the velocity component $v(\tau, r)$:

$$\frac{1}{\mathrm{Re}}(L - M^2)^2\psi - (L - M^2)\frac{\partial\psi}{\partial\tau} = 2M\omega v,$$
$$\frac{1}{\mathrm{Re}}(L - M^2)v - \frac{\partial v}{\partial\tau} = 2aM\psi, \tag{1}$$
$$\psi = \frac{\partial\psi}{\partial r} = v = 0 \quad \text{for} \quad r = 1,\ R_2/\Delta R.$$

The system (1) is written in dimensionless quantities, where $\Delta R = R_2 - R_1$, $\Delta R\Omega_1$, $1/\Omega_1$ are taken as characteristic units for the length, velocity, and time, respectively.

From purely computational consideration (see [6]) it is expedient to replace the system (1) by a system of three second order equations by introducing an auxiliary function according to the relationship

$$(L - M^2)\psi(\tau, r) = \varphi(\tau, r). \tag{2}$$

Using (2), let us write the system (1) in matrix form

$$A\mathbf{X} - B\frac{\partial\mathbf{X}}{\partial\tau} = 0, \quad \mathbf{X} = \mathbf{X}(\psi, \varphi, v),$$

$$A = \begin{bmatrix} L - M^2 & -1 & 0 \\ 0 & \frac{1}{\mathrm{Re}}(L - M^2) & -2M\omega \\ -2Ma & 0 & \frac{1}{\mathrm{Re}}(L - M^2) \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3}$$

under the same boundary conditions.

If we separate variables in (3), we then obtain a problem to determine the eigenvalues in the following form

$$A\mathbf{X} - \lambda B\mathbf{X} = 0, \quad \psi = \frac{d\psi}{dr} = v = 0 \quad \text{for} \quad r = 1,\ R_2/\Delta R, \tag{4}$$

where the components of the vector **X** in this equation are functions only of the coordinate r.

2. If the build-up method is used in the numerical determination of the eigenvectors of the problem (4) according to [6], the finite-difference problem approximating the system (3) can be written as

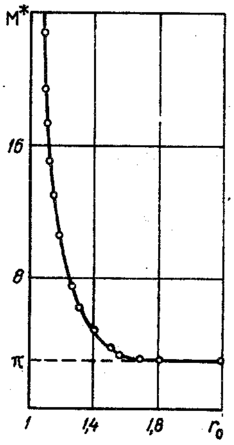

Fig. 2

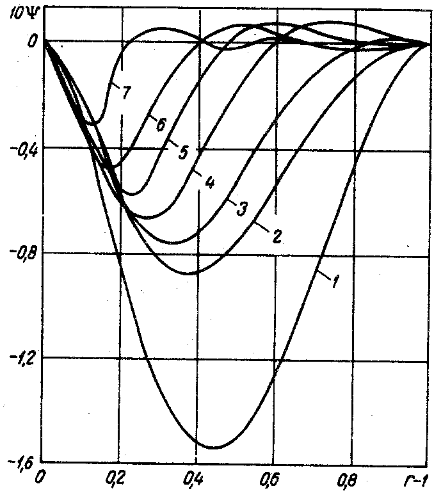

Fig. 3

Fig. 2. Dependence of the critical wave number M* on the radius of the nodal point r_0.

Fig. 3. Amplitude characteristics of the stream function $\psi_1(r)$ of the first eigenvector in the domain of the critical values of the parameters for negative values of β according to Table 1.

$$A_1^{(i)}\mathbf{X}_{i+1}^j + B_1^{(i)}\mathbf{X}_i^j + C_1^{(i)}\mathbf{X}_{i-1}^j = A_2^{(i)}\mathbf{X}_{i+1}^{j-1} + B_2^{(i)}\mathbf{X}_i^{j-1} + C_2^{(i)}\mathbf{X}_{i-1}^{j-1}$$

$$(i = 1,\ 2,\ 3,\ \ldots,\ N-1;\quad N = 1/h;\quad j = 1,\ 2,\ 3,\ \ldots),$$

$$A_1^{(i)} = \varepsilon\left(1 + \frac{h}{2r_i}\right)I,\quad C_1^{(i)} = \varepsilon\left(1 - \frac{h}{2r_i}\right)I,\quad \mathbf{X}_i^j = \mathbf{X}(\psi_i^j,\ \varphi_i^j,\ v_i^j),$$

$$B_1^{(i)} = \begin{bmatrix} \beta_{11} & \beta_{12} & 0 \\ 0 & \beta_{22} & \beta_{23} \\ \beta_{31} & 0 & \beta_{33} \end{bmatrix},\quad \begin{aligned} &\beta_{11} = -\varepsilon\left(2 + \frac{h^2}{r_i^2} + h^2M^2\right),\quad \beta_{12} = -\varepsilon h^2, \\ &\beta_{22} = \beta_{33} = -\mathrm{Re}\frac{h^2}{\Delta\tau} + \beta_{11},\quad \beta_{31} = -2\varepsilon h^2\,\mathrm{Re}\,aM, \\ &\beta_{23} = -2\varepsilon\,\mathrm{Re}\left(a + \frac{b}{r_i^2}\right)Mh^2,\quad r_i = 1 + ih. \end{aligned} \qquad (5)$$

Here h is the coordinate spacing, $\Delta\tau$ is the time spacing, the parameter ε varies on the segment [0, 1], the subscript i refers to a change in the coordinate and j to a change in time, and I is the third-order unit matrix.

The matrices $A_2^{(i)}$, $B_2^{(i)}$, $C_2^{(i)}$ are analogous to the corresponding matrices $A_1^{(i)}$, $B_1^{(i)}$, $C_1^{(i)}$ except the parameter ε should be replaced by $\varepsilon - 1$ in the relations for the elements.

A two-layer, six-point pattern in the time was used to obtain the finite-difference equation (5) with a second order approximation of the differential operators.

Under appropriate boundary conditions (5) is solved by the matrix factorization method. The process of determining successive approximations in the time in conformity with the method used to solve (5) is denoted by the operator A so that

$$\mathbf{X}^j = \tilde{A}\mathbf{X}^{j-1}\quad (j = 1,\ 2,\ 3,\ \ldots). \qquad (6)$$

An invariant subspace extended over a system of three corresponding eigenvectors (see [6], Section 3) is constructed to determine the first three eigenvalues of the finite-difference problem corresponding to (4). The crux of this construction consists briefly of the following. A sequence of a system of three mutually orthogonal vectors **X** (k = 1, 2, 3; j = 1, 2, 3, . . .) is constructed by choosing three arbitrary vectors $\mathbf{X}_b^0$ (k = 1, 2, 3) (linearly-independent desirably) of the dimensionality 3(N + 1) each.

Orthogonalization and normalization of the vectors during the calculation are performed in the generalized sense by means of the relationship

TABLE 1. Values of the Parameters of the Problem for which the Curves $\psi_1(r)$ are Computed in Fig. 3

Curve number	β	M	Re	λ	r_0-1
1	0	3,16	68,2	0,001	1
2	—0,261	3,41	75,7	—0,021	0,57
3	—0,342	3,76	87,5	—0,007	0,51
4	—0,52	4,92	118	0,002	0,41
5	—0,75	6,20	158	0,009	0,32
6	—1,0	7,54	205	—0,021	0,27
7	—2,0	13,0	433	0,003	0,16

$$N_0 = -\sum_{k=0}^{N} [\psi_n(r_k)\varphi_m(r_k) - v_n(r_k)v_m(r_k)]\, r_k h \quad (n,\ m = 1,\ 2,\ 3). \tag{7}$$

The vector sequence constructed in this manner as $j \to \infty$ has the limits X_k (k = 1, 2, 3) which lie in the above-mentioned invariant subspace. Taking these limit vectors as the basis of the given subspace, a matrix $L_0\{\alpha_{mn}\}$ of the induced operator is constructed by means of the expression

$$\alpha_{mn} = (\tilde{A}X_n,\ X_m) \quad (n,\ m = 1,\ 2,\ 3). \tag{8}$$

It is understood than in numerical computations according to (8), the approximations $\tilde{X}_n^j$ are taken as $\tilde{X}_n$ under the condition required for the stabilization of the iteration process (6).

The eigenvalues of the matrix L_0 agree with the eigenvalues of the parameter q_k (k = 1, 2, 3) of the problem (5), which are related to the desired eigenvalues λ_k (k = 1, 2, 3) of the finite-difference problem (4) by means of the known expression

$$\lambda_k = \frac{q_k - 1}{(1 - \varepsilon + \varepsilon q_k)\,\Delta\tau} \quad (k = 1,\ 2,\ 3). \tag{9}$$

The accuracy of determining the eigenvectors is checked by the norm of the residual δ_k (k = 1, 2, 3) of the last and the preceding time-approximations of these vectors

$$\delta_k = \sqrt{(\vec{\Delta}_k,\ \vec{\Delta}_k)},\quad \vec{\Delta}_k = \tilde{A}U_k^j - q_k U_k^j \quad (k=1,\ 2,\ 3;\ \ j = 1,\ 2,\ 3,\ldots), \tag{10}$$

where U_k^j are eigenvectors of the problem representing linearly-independent combinations of the approximations and X_k^j, where the components of the corresponding eigenvectors of the matrix L_0 are constants in these combinations.

3. In conformity with the conditions at which the Donelly experiments were conducted, let us perform a computation for the ratio of the cylinder radii $R_2/R_1 = 2$. In this case the parameters in the expression for the angular velocity ω are determined by means of the relations

$$a = \frac{4\beta - 1}{3};\quad b = -\frac{4(\beta - 1)}{3};\quad \beta = \frac{\Omega_2}{\Omega_1}. \tag{11}$$

For cylinders rotating in one direction it is sufficient to consider the change in the angular parameter β in the range $0 \le \beta \le 0.25$ in computations of the stability curve in conformity with the Rayleigh stability criterion for the inviscid case.

When the cylinders rotate in opposite directions, the velocity of the steady flow of the fluid layer at some point along the radius changes sign. The point at which the velocity distribution curve passes through zero is called a nodal point. The radius of this point is determined by the formula

$$r_0 = \sqrt{-\frac{b}{a}} = 2\sqrt{\frac{\beta - 1}{4\beta - 1}}. \tag{12}$$

The sequence in the calculations to construct the stability line (Figs. 1a and b) and the other characteristics of the problem (Figs. 2 and 3) reduces briefly to the following. For a given value of the angular parameter β the neutral perturbation curve Re = f(M) is computed. The critical value of the Reynolds criterion Re* and its corresponding wave number M* are determined from the results of this dependence by quadratic interpolation.

It is seen from Fig. 1 that for $\beta > 0$ the stability line asymptotically approaches the line of the Rayleigh criterion for the inviscid case. Also presented in Fig. 1b, in addition to the computed curve, is a curve of the experimental stability dependence according to the data in [3] for $\beta < 0$. It is seen that prior to the values $-\Omega_2/\nu \sim 20$ the experimental and theoretical curves agree. This domain is shown in Fig. 1 in a smaller scale. Let us note that in this domain the results we obtained also agree completely with the results in [2] in which the stability curve for $\beta < 0$ is computed up to the value $\Omega_2/\nu = -6$. The stability curve we obtained for the values $-\Omega_2/\nu > 20$ lies below the experimental curve.

The nature of the stable secondary flows is described by the first eigenvector found for values of the external parameters characterizing the neutral perturbations. Presented in Fig. 3 are the amplitude characteristics of the stream functions as components of the first eigenvector for different values of the angular parameter β in the range of critical values.

For positive values of β the critical value of the wave number M^* is independent of β and equals ~ 3.15 (Fig. 2). In this case the stable vortex cores occupy the whole gap between the cylinders and the vertical spacing between the vortices (half the wavelength along the z axis equals π/M^*) equals the gap width ΔR. This is verified well also by the nature of the dependence $\psi_1(r)$ of the first eigenvector (see Fig. 3, curve 1 for $\beta = 0$, the picture will be analogous for other values $\beta > 0$).

For the cylinders rotating in opposite directions ($\beta < 0$) the nature of the stable secondary flows should be complicated since two steady flow zones originate in the gap between the cylinders in this case, whose properties are not identical relative to small perturbations. Stable secondary flows can originate in the first zone between the inner cylinder and the nodal point, for definite values of the external parameters. All the perturbations originating in conformity with the Rayleigh stability condition for the inviscid case should damp out in the second zone between the nodal point and the outer cylinder. Indeed, as computations show (Fig. 3), the picture of the possible stable secondary flows is not part of the diagram presented above.

An analysis of the curves in Fig. 3 shows that for small absolute value of β the secondary flow vortex is propagated still more on both zones, being noticeably attenuated in the direction to the outer cylinder (this can be estimated by the slope of the tangent to the curve). The height of the core along the z axis remains approximately equal to the gap width, as before, for these values of β.

As the absolute value of β grows, the domain of vortex localization becomes less than the gap width although it still remains somewhat greater than the nodal distance. At the same time, an additional vortex of considerably lesser intensity and opposite direction of rotation (curves 3-5) appears behind the first vortex (called the principal vortex) which encloses the first and part of the second zones. For $\beta = -1$ already two additional vortices originate behind the principal vortex, where the directions of vortex rotation change successfully to the opposite (curve 6). For $\beta = -2$ the number of additional vortices increases although their intensity (with the exception of the first additional vortex) is practically zero relative to the principal vortex (curve 7).

Therefore, as the cylinders rotate in opposite directions, starting with some value of the angular parameter β, the stable secondary flows are characterized by a multicored structure in the radial direction. The height of the cores of this structure along the z axis is determined in conformity with the magnitude of the wave number M^* (Fig. 2).

NOTATION

$R_1, R_2, \Omega_1, \Omega_2$	are the radii and angular velocities of the inner and outer cylinders, respectively;
$\Delta R = R_2 - R_1$	is the width of the gap between the cylinders;
$r = R/\Delta R$	is the dimensionless radius;
r_0	is the dimensionless radius of the nodal point;
$Re = \Omega_1(\Delta R)^2/\nu$	is the Reynolds criterion;
ν	is the coefficient of kinematic viscosity;
$\beta = \Omega_2/\Omega_1$	is the angular parameter;
$\omega = a + b/r^2$	is the angular velocity distribution of steady fluid motion, where

$$a = \frac{\gamma\beta - 1}{\gamma - 1},\quad b = -\frac{\gamma(\beta - 1)}{\gamma - 1},\quad \gamma = \left(\frac{R_2}{R_1}\right)^2;\quad L = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}.$$

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